

Optimization and Decision Making

2020

Time allowed: 2 hours

*Full marks may be obtained for correct answers to
THREE of the FOUR questions.
All necessary working must be shown.*

1. Suppose that you have to choose an optimal portfolio from a list of n stocks. Stock i has expected revenue rate μ_i with variance σ_i^2 for $i = 1, \dots, n$, and the covariance of the revenues of stocks i and j is given by σ_{ij} for $i \neq j, i, j = 1, \dots, n$. The proportion of stock i in the portfolio is denoted by w_i .
 - (a) The expected revenue from the portfolio is $\sum_{i=1}^n w_i \mu_i$. Show that the variance of the portfolio revenue is given by $\sigma_P^2 = \sum_{i,j=1}^n w_i w_j \sigma_{ij}$. [5]
 - (b) Formulate the general n stock problem using Lagrange multipliers, and hence write down a set of equations for the optimal value of the w_i s. Hence use this set of equations to show that the optimal solution for the 3 stock problem is given by the following equations:

$$\begin{aligned}
 & [\mu_2(\sigma_1^2 - \sigma_{13}) + \mu_1(\sigma_{13} - \sigma_{12}) + \mu_3(\sigma_{12} - \sigma_1^2)]w_1 + \\
 & [\mu_3(\sigma_2^2 - \sigma_{12}) + \mu_2(\sigma_{12} - \sigma_{23}) + \mu_1(\sigma_{23} - \sigma_2^2)]w_2 + \\
 & [\mu_1(\sigma_3^2 - \sigma_{23}) + \mu_3(\sigma_{23} - \sigma_{13}) + \mu_2(\sigma_{13} - \sigma_3^2)]w_3 = 0, \\
 & w_2 = \frac{(\mu_P - \mu_3) - w_1(\mu_1 - \mu_3)}{\mu_2 - \mu_3}, \\
 & w_3 = \frac{(\mu_2 - \mu_P) + w_1(\mu_1 - \mu_2)}{\mu_2 - \mu_3}.
 \end{aligned}$$

[10]

- (c) Now consider a problem with three stocks where the means, variances and covariances are as follows:
 $\mu_1 = 0.05, \mu_2 = 0.06, \mu_3 = 0.09, \sigma_1^2 = 0.2, \sigma_2^2 = 0.2, \sigma_3^2 = 1.2, \sigma_{12} = 0.1, \sigma_{13} = -0.2$ and $\sigma_{23} = 0.1$.
 Find the optimal portfolio (i.e. the one with the minimum variance) if the targeted expected rate of return is 0.06. [10]
- (d) Suppose that the targeted expected rate of return now changes to 0.07. What is the optimal portfolio now? [5]

2. (a) In the context of assignment problems, state *König's Theorem*. Describe in detail the *Hungarian method* to find the optimal assignment of a problem. [9]
- (b) A manager has a team of four, and must assign appropriate jobs to each, depending upon their abilities. She carries out an assessment test to help with this process, which assigns scores S for their abilities in each role as follows (where higher scores are better):
 Team member A: role 1 $S=90$, role 2 $S=80$, role 3 $S=70$, role 4 $S=85$.
 Team member B: role 1 $S=70$, role 2 $S=55$, role 3 $S=40$, role 4 $S=65$.
 Team member C: role 1 $S=75$, role 2 $S=85$, role 3 $S=45$, role 4 $S=60$.
 Team member D: role 1 $S=55$, role 2 $S=65$, role 3 $S=40$, role 4 $S=45$.
 Find the optimal assignment(s) for this problem. [15]
- (c) Suppose that her manager insisted that team member B be given role 3. Formulate (without solving) the new problem. [3]
- (d) Suppose instead that her manager had explicitly banned team member B from performing role 3. Formulate this new problem (again without solving). [3]
3. A luxury furniture manufacturer makes four different types of chair; Dining Chairs, Armchairs, Reclining Chairs and Sofas. There are different materials, but the only one in short supply is locally produced wood which is the only wood they use. The manufacturer has broken down the requirements into two for each chair, Labour and Wood. The table below lists the four types, the Labour and Wood required for each chair, the total current (weekly) supply of Labour and Wood to the manufacturer and the profit per chair.

Chair type	Dining	Armchair	Reclining	Sofa	Supply
Labour (hours)	2	4	8	10	4000
Wood (kg)	8	12	12	16	8000
Profit (pounds)	80	120	160	200	

- (a) Set up the problem as a linear program. [12]
- (b) Solve the problem using the simplex method. [15]
- (c) Suppose that the company directors insisted on no single type of product providing more than 40% of the items produced. Formulate the new linear programme for this problem (you do not have to solve it). [3]

4. A construction company must decide whether to invest money in buying a derelict commercial building and potentially developing it into residential accommodation. It estimates that there is a 60% chance that permission for a change of use will be obtained to develop the site within four years. If permission has not been obtained at the end of this period, the company would abandon the project and resell the site, with a loss of £5 million. If permission has been obtained, the type of development would have to be decided upon, Houses or Flats. The returns generated would depend upon the strength of the economy, categorized as either Strong or Weak. If the company opted for Flats and demand was Strong, a £8 million profit would be obtained. For Weak demand, however, there would be a loss of £3 million. If Houses are chosen, Strong demand would generate a profit of £5 million and Weak demand would generate a profit of £1 million. The company estimates that there is a 70% chance of Strong demand.
- Construct a decision tree to represent this decision problem. [13]
 - Assuming that the company's objective is to maximize its expected profit, determine the policy that it should adopt. [4]
 - The company's estimated probability of Strong demand may be wrong. Assuming that all other problem elements remain the same, find the course of action for all possible probabilities of Strong demand. [4]
 - For the original problem in (b), Suppose that the probability of planning permission being granted is not fixed, but depends upon whether demand will be Strong or Weak, with there being a 80% chance of permission being granted if there is Strong demand, but only a 50% chance for Weak demand. What is the optimal policy now? [5]
 - Before the final decision is taken the company is taken over by a new owner, who has utilities shown below for the sums of money involved, and will make the decision using these. What implications does this have for the policy that you found from b)? [4]

Value (£ million)	-5	-3	0	1	5	8
Utility	0	0.25	0.5	0.75	0.9	1

Internal Examiner: Professor M. Broom
 External Examiner: Professor D. Gounopoulos